

# Linear dilatation structures and inverse semigroups

Marius Buliga

Institute of Mathematics, Romanian Academy  
P.O. BOX 1-764, RO 014700  
Bucureşti, Romania  
Marius.Buliga@imar.ro

This version: 07.06.2007

## Abstract

A dilatation structure encodes the approximate self-similarity of a metric space. A metric space  $(X, d)$  which admits a strong dilatation structure (definition 2.2) has a metric tangent space at any point  $x \in X$  (theorem 4.1), and any such metric tangent space has an algebraic structure of a conical group (theorem 4.2). Particular examples of conical groups are Carnot groups: these are simply connected Lie groups whose Lie algebra admits a positive graduation.

The dilatation structures associated to conical (or Carnot) groups are linear, in the sense of definition 5.3. Thus conical groups are the right generalization of normed vector spaces, from the point of view of dilatation structures.

We prove that for dilatation structures linearity is equivalent to a statement about the inverse semigroup generated by the family of dilatations forming a dilatation structure on a metric space.

The result is new for Carnot groups and the proof seems to be new even for the particular case of normed vector spaces.

**Keywords:** inverse semigroups, Carnot groups, dilatation structures

**MSC classes:** 20M18; 22E20; 20F65

## 1 Inverse semigroups and Menelaos theorem

**Definition 1.1** *A semigroup  $S$  is an inverse semigroup if for any  $x \in S$  there is an unique element  $x^{-1} \in S$  such that  $xx^{-1}x = x$  and  $x^{-1}xx^{-1} = x^{-1}$ .*

An important example of an inverse semigroup is  $I(X)$ , the class of all bijective maps  $\phi : \text{dom } \phi \rightarrow \text{im } \phi$ , with  $\text{dom } \phi, \text{im } \phi \subset X$ . The semigroup operation is the composition of functions in the largest domain where this makes sense.

By the Vagner-Preston representation theorem [6] every inverse semigroup is isomorphic to a subsemigroup of  $I(X)$ , for some set  $X$ .

## 1.1 A toy example

Let  $(\mathbb{V}, \|\cdot\|)$  be a finite dimensional, normed, real vector space. By definition the dilatation based at  $x$ , of coefficient  $\varepsilon > 0$ , is the function

$$\delta_\varepsilon^x : \mathbb{V} \rightarrow \mathbb{V} \quad , \quad \delta_\varepsilon^x y = x + \varepsilon(-x + y) \quad .$$

For fixed  $x$  the dilatations based at  $x$  form a one parameter group which contracts any bounded neighbourhood of  $x$  to a point, uniformly with respect to  $x$ .

With the distance  $d$  induced by the norm, the metric space  $(\mathbb{V}, d)$  is complete and locally compact. For any  $x \in \mathbb{V}$  and any  $\varepsilon > 0$  the distance  $d$  behaves well with respect to the dilatation  $\delta_\varepsilon^x$  in the sense: for any  $u, v \in \mathbb{V}$  we have

$$\frac{1}{\varepsilon} d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) = d(u, v) \quad . \quad (1.1.1)$$

Dilatations encode much more than the metric structure of the space  $(\mathbb{V}, d)$ . Indeed, we can reconstruct the algebraic structure of the vector space  $\mathbb{V}$  from dilatations. For example let us define for any  $x, u, v \in \mathbb{V}$  and  $\varepsilon > 0$ :

$$\Sigma_\varepsilon^x(u, v) = \delta_{\varepsilon^{-1}}^x \delta_\varepsilon^x u(v) \quad .$$

A simple computation shows that  $\Sigma_\varepsilon^x(u, v) = u + \varepsilon(-u + x) + (-x + v)$ , therefore we can recover the addition operation in  $\mathbb{V}$  by using the formula:

$$\lim_{\varepsilon \rightarrow 0} \Sigma_\varepsilon^x(u, v) = u + (-x + v) \quad . \quad (1.1.2)$$

This is the addition operation translated such that the neutral element is  $x$ . Thus, for  $x = 0$ , we recover the usual addition operation.

Affine continuous transformations  $A : \mathbb{V} \rightarrow \mathbb{V}$  admit the following description in terms of dilatations. A continuous transformation  $A : \mathbb{V} \rightarrow \mathbb{V}$  is affine if and only if for any  $\varepsilon \in (0, 1)$ ,  $x, y \in \mathbb{V}$  we have

$$A \delta_\varepsilon^x y = \delta_\varepsilon^{Ax} Ay \quad . \quad (1.1.3)$$

Any dilatation is an affine transformation, hence for any  $x, y \in \mathbb{V}$  and  $\varepsilon, \mu > 0$  we have

$$\delta_\mu^y \delta_\varepsilon^x = \delta_\varepsilon^{\delta_\mu^y x} \delta_\mu^y \quad . \quad (1.1.4)$$

Moreover, some compositions of dilatations are dilatations. This is precisely stated in the next theorem, which is equivalent with the Menelaos theorem in euclidean geometry.

**Theorem 1.2** *For any  $x, y \in \mathbb{V}$  and  $\varepsilon, \mu > 0$  such that  $\varepsilon\mu \neq 1$  there exists an unique  $w \in \mathbb{V}$  such that*

$$\delta_\mu^y \delta_\varepsilon^x = \delta_{\varepsilon\mu}^w \quad .$$

For the proof see Artin [1]. A straightforward consequence of this theorem is the following result.

**Corollary 1.3** *The inverse subsemigroup of  $I(\mathbb{V})$  generated by dilatations of the space  $\mathbb{V}$  is made of all dilatations and all translations in  $\mathbb{V}$ .*

**Proof.** Indeed, by theorem 1.2 a composition of two dilatations with coefficients  $\varepsilon, \mu$  with  $\varepsilon\mu \neq 1$  is a dilatation. By direct computation, if  $\varepsilon\mu = 1$  then we obtain translations. This is in fact compatible with (1.1.2), but is a stronger statement, due to the fact that dilatations are affine in the sense of relation (1.1.4).

Moreover any translation can be expressed as a composition of two dilatations with coefficients  $\varepsilon, \mu$  such that  $\varepsilon\mu = 1$ . Finally, any composition between a translation and a dilatation is again a dilatation.  $\square$

## 1.2 Focus on dilatations

Suppose that we take the dilatations as basic data for the toy example above. Namely, instead of giving to the space  $\mathbb{V}$  a structure of real, normed vector space, we give only the distance  $d$  and the dilatations  $\delta_\varepsilon^x$  for all  $x \in X$  and  $\varepsilon > 0$ . We should add some relations which prescribe:

- the behaviour of the distance with respect to dilatations, for example some form of relation (1.1.1),
- the interaction between dilatations, for example the existence of the limit from the left hand side of relation (1.1.2).

We denote such a collection of data by  $(\mathbb{V}, d, \delta)$  and call it a dilatation structure (see further definition 2.2).

In this paper we ask if there is any relationship between dilatations and inverse semigroups, generalizing relation (1.1.4) and corollary 1.3.

Dilatation structures are far more general than our toy example. A dilatation structure on a metric space, introduced in [3], is a notion in between a group and a differential structure, expressing the approximate self-similarity of the metric space where it lives.

A metric space  $(X, d)$  which admits a strong dilatation structure (definition 2.2) has a metric tangent space at any point  $x \in X$  (theorem 4.1), and any such metric tangent space has an algebraic structure of a conical group (theorem 4.2). Conical groups are particular examples of contractible groups. An important class of conical groups is formed by Carnot groups: these are simply connected Lie groups whose Lie algebra admits a positive graduation. Carnot groups appear in many situations, in particular in relation with sub-riemannian geometry cf. Bellaïche [2], groups with polynomial growth cf. Gromov [5], or Margulis type rigidity results cf. Pansu [7].

The dilatation structures associated to conical (or Carnot) groups are linear, in the sense of relation (1.1.4), see also definition 5.3. We actually proved in [4] (here theorem 5.4) that a linear dilatation structure always comes from some associated conical group. Thus conical groups are the right generalization of normed vector spaces, from the point of view of dilatation structures.

## 2 Dilatation structures

We present here an introduction into the subject of dilatation structures, following Buliga [3].

### 2.1 Notations

Let  $\Gamma$  be a topological separated commutative group endowed with a continuous group morphism

$$\nu : \Gamma \rightarrow (0, +\infty)$$

with  $\inf \nu(\Gamma) = 0$ . Here  $(0, +\infty)$  is taken as a group with multiplication. The neutral element of  $\Gamma$  is denoted by 1. We use the multiplicative notation for the operation in  $\Gamma$ .

The morphism  $\nu$  defines an invariant topological filter on  $\Gamma$  (equivalently, an end). Indeed, this is the filter generated by the open sets  $\nu^{-1}(0, a)$ ,  $a > 0$ . From now on we shall name this topological filter (end) by "0" and we shall write  $\varepsilon \in \Gamma \rightarrow 0$  for  $\nu(\varepsilon) \in (0, +\infty) \rightarrow 0$ .

The set  $\Gamma_1 = \nu^{-1}(0, 1]$  is a semigroup. We note  $\bar{\Gamma}_1 = \Gamma_1 \cup \{0\}$ . On the set  $\bar{\Gamma} = \Gamma \cup \{0\}$  we extend the operation on  $\Gamma$  by adding the rules  $00 = 0$  and  $\varepsilon 0 = 0$  for any  $\varepsilon \in \Gamma$ . This is in agreement with the invariance of the end 0 with respect to translations in  $\Gamma$ .

The space  $(X, d)$  is a complete, locally compact metric space. For any  $r > 0$  and any  $x \in X$  we denote by  $B(x, r)$  the open ball of center  $x$  and radius  $r$  in the metric space  $X$ .

On the metric space  $(X, d)$  we work with the topology (and uniformity) induced by the distance. For any  $x \in X$  we denote by  $\mathcal{V}(x)$  the topological filter of open neighbourhoods of  $x$ .

### 2.2 Axioms of dilatation structures

The first axiom is a preparation for the next axioms. That is why we counted it as axiom 0.

**A0.** The dilatations

$$\delta_\varepsilon^x : U(x) \rightarrow V_\varepsilon(x)$$

are defined for any  $\varepsilon \in \Gamma, \nu(\varepsilon) \leq 1$ . The sets  $U(x), V_\varepsilon(x)$  are open neighbourhoods of  $x$ . All dilatations are homeomorphisms (invertible, continuous, with continuous inverse).

We suppose that there is a number  $1 < A$  such that for any  $x \in X$  we have

$$\bar{B}_d(x, A) \subset U(x) .$$

We suppose that for all  $\varepsilon \in \Gamma$ ,  $\nu(\varepsilon) \in (0, 1)$ , we have

$$B_d(x, \nu(\varepsilon)) \subset \delta_\varepsilon^x B_d(x, A) \subset V_\varepsilon(x) \subset U(x) .$$

There is a number  $B \in (1, A]$  such that for any  $\varepsilon \in \Gamma$  with  $\nu(\varepsilon) \in (1, +\infty)$  the associated dilatation

$$\delta_\varepsilon^x : W_\varepsilon(x) \rightarrow B_d(x, B) ,$$

is injective, invertible on the image. We shall suppose that  $W_\varepsilon(x) \in \mathcal{V}(x)$ , that  $V_{\varepsilon^{-1}}(x) \subset W_\varepsilon(x)$  and that for all  $\varepsilon \in \Gamma_1$  and  $u \in U(x)$  we have

$$\delta_{\varepsilon^{-1}}^x \delta_\varepsilon^x u = u .$$

We have therefore the following string of inclusions, for any  $\varepsilon \in \Gamma$ ,  $\nu(\varepsilon) \leq 1$ , and any  $x \in X$ :

$$B_d(x, \nu(\varepsilon)) \subset \delta_\varepsilon^x B_d(x, A) \subset V_\varepsilon(x) \subset W_{\varepsilon^{-1}}(x) \subset \delta_\varepsilon^x B_d(x, B) .$$

A further technical condition on the sets  $V_\varepsilon(x)$  and  $W_\varepsilon(x)$  will be given just before the axiom A4. (This condition will be counted as part of axiom A0.)

**A1.** We have  $\delta_\varepsilon^x x = x$  for any point  $x$ . We also have  $\delta_1^x = id$  for any  $x \in X$ .

Let us define the topological space

$$\begin{aligned} \text{dom } \delta = \{(\varepsilon, x, y) \in \Gamma \times X \times X : & \quad \text{if } \nu(\varepsilon) \leq 1 \text{ then } y \in U(x) , \\ & \text{else } y \in W_\varepsilon(x)\} \end{aligned}$$

with the topology inherited from the product topology on  $\Gamma \times X \times X$ . Consider also  $Cl(\text{dom } \delta)$ , the closure of  $\text{dom } \delta$  in  $\bar{\Gamma} \times X \times X$  with product topology. The function  $\delta : \text{dom } \delta \rightarrow X$  defined by  $\delta(\varepsilon, x, y) = \delta_\varepsilon^x y$  is continuous. Moreover, it can be continuously extended to  $Cl(\text{dom } \delta)$  and we have

$$\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon^x y = x .$$

**A2.** For any  $x \in K$ ,  $\varepsilon, \mu \in \Gamma_1$  and  $u \in \bar{B}_d(x, A)$  we have:

$$\delta_\varepsilon^x \delta_\mu^x u = \delta_{\varepsilon\mu}^x u .$$

**A3.** For any  $x$  there is a function  $(u, v) \mapsto d^x(u, v)$ , defined for any  $u, v$  in the closed ball (in distance  $d$ )  $\bar{B}_d(x, A)$ , such that

$$\lim_{\varepsilon \rightarrow 0} \sup \left\{ \left| \frac{1}{\varepsilon} d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) - d^x(u, v) \right| : u, v \in \bar{B}_d(x, A) \right\} = 0$$

uniformly with respect to  $x$  in compact set.

**Remark 2.1** The "distance"  $d^x$  can be degenerated: there might exist  $v, w \in U(x)$  such that  $d^x(v, w) = 0$ .

For the following axiom to make sense we impose a technical condition on the co-domains  $V_\varepsilon(x)$ : for any compact set  $K \subset X$  there are  $R = R(K) > 0$  and  $\varepsilon_0 = \varepsilon(K) \in (0, 1)$  such that for all  $u, v \in \bar{B}_d(x, R)$  and all  $\varepsilon \in \Gamma$ ,  $\nu(\varepsilon) \in (0, \varepsilon_0)$ , we have

$$\delta_\varepsilon^x v \in W_{\varepsilon^{-1}}(\delta_\varepsilon^x u) .$$

With this assumption the following notation makes sense:

$$\Delta_\varepsilon^x(u, v) = \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v .$$

The next axiom can now be stated:

**A4.** We have the limit

$$\lim_{\varepsilon \rightarrow 0} \Delta_\varepsilon^x(u, v) = \Delta^x(u, v)$$

uniformly with respect to  $x, u, v$  in compact set.

**Definition 2.2** A triple  $(X, d, \delta)$  which satisfies A0, A1, A2, A3, but  $d^x$  is degenerate for some  $x \in X$ , is called *degenerate dilatation structure*.

If the triple  $(X, d, \delta)$  satisfies A0, A1, A2, A3 and  $d^x$  is non-degenerate for any  $x \in X$ , then we call it a *dilatation structure*.

If a dilatation structure satisfies A4 then we call it *strong dilatation structure*.

### 3 Normed conical groups

We shall need further the notion of normed conical group. Motivated by the case of a Lie group endowed with a Carnot-Carathéodory distance induced by a left invariant distribution, we shall use the following definition of a local uniform group.

Let  $G$  be a group. We introduce first the double of  $G$ , as the group  $G^{(2)} = G \times G$  with operation

$$(x, u)(y, v) = (xy, y^{-1}uyv) .$$

The operation on the group  $G$ , seen as the function  $op : G^{(2)} \rightarrow G$ ,  $op(x, y) = xy$  is a group morphism. Also the inclusions:

$$i' : G \rightarrow G^{(2)} , \quad i'(x) = (x, e)$$

$$i'' : G \rightarrow G^{(2)} , \quad i''(x) = (x, x^{-1})$$

are group morphisms.

**Definition 3.1** 1.  $G$  is an *uniform group* if we have two uniformity structures, on  $G$  and  $G \times G$ , such that  $op$ ,  $i'$ ,  $i''$  are uniformly continuous.

2. A local action of a uniform group  $G$  on a uniform pointed space  $(X, x_0)$  is a function  $\phi \in W \in \mathcal{V}(e) \mapsto \hat{\phi} : U_\phi \in \mathcal{V}(x_0) \rightarrow V_\phi \in \mathcal{V}(x_0)$  such that:

- (a) the map  $(\phi, x) \mapsto \hat{\phi}(x)$  is uniformly continuous from  $G \times X$  (with product uniformity) to  $X$ ,
- (b) for any  $\phi, \psi \in G$  there is  $D \in \mathcal{V}(x_0)$  such that for any  $x \in D$   $\phi\psi^{-1}(x)$  and  $\hat{\phi}(\hat{\psi}^{-1}(x))$  make sense and  $\phi\hat{\psi}^{-1}(x) = \hat{\phi}(\hat{\psi}^{-1}(x))$ .

3. Finally, a local group is an uniform space  $G$  with an operation defined in a neighbourhood of  $(e, e) \subset G \times G$  which satisfies the uniform group axioms locally.

**Definition 3.2** A normed (local) conical group  $(G, \delta, \|\cdot\|)$  is (local) group endowed with: (I) a (local) action of  $\Gamma$  by morphisms  $\delta_\varepsilon$  such that  $\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon x = e$  for any  $x$  in a neighbourhood of the neutral element  $e$ ; (II) a continuous norm function  $\|\cdot\| : G \rightarrow \mathbb{R}$  which satisfies (locally, in a neighbourhood of the neutral element  $e$ ) the properties:

- (a) for any  $x$  we have  $\|x\| \geq 0$ ; if  $\|x\| = 0$  then  $x = e$ ,
- (b) for any  $x, y$  we have  $\|xy\| \leq \|x\| + \|y\|$ ,
- (c) for any  $x$  we have  $\|x^{-1}\| = \|x\|$ ,
- (d) for any  $\varepsilon \in \Gamma$ ,  $\nu(\varepsilon) \leq 1$  and any  $x$  we have  $\|\delta_\varepsilon x\| = \nu(\varepsilon) \|x\|$ .

Particular cases of normed conical groups are:

- Carnot groups, that is simply connected real Lie groups whose Lie algebra admits a positive graduation,
- nilpotent p-adic groups admitting a contractive automorphism.

A very particular case of a normed conical group is described in the toy example: to any real, finite dimensional, normed vector space  $\mathbb{V}$  we may associate the normed conical group  $(\mathbb{V}, +, \delta, \|\cdot\|)$ , with dilatations  $\delta$  previously described.

In a normed conical group  $(G, \delta)$  we define dilatations based in any point  $x \in G$  by

$$\delta_\varepsilon^x u = x \delta_\varepsilon (x^{-1} u). \quad (3.0.1)$$

There is also a natural left invariant distance given by

$$d(x, y) = \|x^{-1} y\| \quad . \quad (3.0.2)$$

The following result is theorem 15 [3].

**Theorem 3.3** Let  $(G, \delta, \|\cdot\|)$  be a locally compact normed group with dilatations. Then  $(G, \delta, d)$  is a strong dilatation structure, where  $\delta$  are the dilatations defined by (3.0.1) and the distance  $d$  is induced by the norm as in (3.0.2).

## 4 Properties of dilatation structures

The following two theorems describe the most important metric and algebraic properties of a dilatation structure. As presented here these are condensed statements, available in full length as theorems 7, 8, 10 in [3].

**Theorem 4.1** *Let  $(X, d, \delta)$  be a dilatation structure. Then the metric space  $(X, d)$  admits a metric tangent space at  $x$ , for any point  $x \in X$ . More precisely we have the following limit:*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \sup \{ |d(u, v) - d^x(u, v)| : d(x, u) \leq \varepsilon, d(x, v) \leq \varepsilon \} = 0 .$$

**Theorem 4.2** *Let  $(X, d, \delta)$  be a strong dilatation structure. Then for any  $x \in X$  the triple  $(U(x), \Sigma^x, \delta^x, d^x)$  is a normed local conical group. This means:*

- (a)  $\Sigma^x$  is a local group operation on  $U(x)$ , with  $x$  as neutral element and  $\text{inv}^x$  as the inverse element function;
- (b) the distance  $d^x$  is left invariant with respect to the group operation from point (a);
- (c) For any  $\varepsilon \in \Gamma$ ,  $\nu(\varepsilon) \leq 1$ , the dilatation  $\delta_\varepsilon^x$  is an automorphism with respect to the group operation from point (a);
- (d) the distance  $d^x$  has the cone property with respect to dilatations: for any  $u, v \in X$  such that  $d(x, u) \leq 1$  and  $d(x, v) \leq 1$  and all  $\mu \in (0, A)$  we have:

$$d^x(u, v) = \frac{1}{\mu} d^x(\delta_\mu^x u, \delta_\mu^x v) .$$

The conical group  $(U(x), \Sigma^x, \delta^x)$  can be regarded as the tangent space of  $(X, d, \delta)$  at  $x$ .

By using proposition 5.4 [8] and from some topological considerations we deduce the following characterisation of tangent spaces associated to some dilatation structures. The following is corollary 4.7 [4].

**Corollary 4.3** *Let  $(X, d, \delta)$  be a dilatation structure with group  $\Gamma = (0, +\infty)$  and the morphism  $\nu$  equal to identity. Then for any  $x \in X$  the local group  $(U(x), \Sigma^x)$  is locally a simply connected Lie group whose Lie algebra admits a positive graduation (a Carnot group).*



## 5 Linearity and dilatation structures

In this section we describe the notion of linearity for dilatation structures, as in Buliga [4].

**Definition 5.1** *Let  $(X, d, \delta)$  be a dilatation structure. A transformation  $A : X \rightarrow X$  is linear if it is Lipschitz and it commutes with dilatations in the following sense: for any  $x \in X$ ,  $u \in U(x)$  and  $\varepsilon \in \Gamma$ ,  $\nu(\varepsilon) < 1$ , if  $A(u) \in U(A(x))$  then*

$$A\delta_\varepsilon^x = \delta^{A(x)} A(u) \quad .$$

In the particular case of  $X$  finite dimensional real, normed vector space,  $d$  the distance given by the norm,  $\Gamma = (0, +\infty)$  and dilatations  $\delta_\varepsilon^x u = x + \varepsilon(u - x)$ , a linear transformations in the sense of definition 5.1 is an affine transformation of the vector space  $X$ . More generally, linear transformations in the sense of definition 5.1 have the expected properties related to linearity, as explained in section 5 [4].

**Convention 5.2** *Further we shall say that a property  $\mathcal{P}(x_1, x_2, x_3, \dots)$  holds for  $x_1, x_2, x_3, \dots$  sufficiently closed if for any compact, non empty set  $K \subset X$ , there is a positive constant  $C(K) > 0$  such that  $\mathcal{P}(x_1, x_2, x_3, \dots)$  is true for any  $x_1, x_2, x_3, \dots \in K$  with  $d(x_i, x_j) \leq C(K)$ .*

For example, the expressions

$$\delta_\varepsilon^x \delta_\mu^y z \quad , \quad \delta_{\mu}^{\delta_\varepsilon^x y} \delta_\varepsilon^x z$$

are well defined for any  $x, y, z \in X$  sufficiently closed and for any  $\varepsilon, \mu \in \Gamma$  with  $\nu(\varepsilon), \nu(\mu) \in (0, 1]$ . Indeed, let  $K \subset X$  be compact, non empty set. Then there is a constant  $C(K) > 0$ , depending on the set  $K$  such that for any  $\varepsilon, \mu \in \Gamma$  with  $\nu(\varepsilon), \nu(\mu) \in (0, 1]$  and any  $x, y, z \in K$  with  $d(x, y), d(x, z), d(y, z) \leq C(K)$  we have

$$\delta_\mu^y z \in V_\varepsilon(x) \quad , \quad \delta_\varepsilon^x z \in V_\mu(\delta_\varepsilon^x y) \quad .$$

Indeed, this is coming from the uniform (with respect to  $K$ ) estimates:

$$d(\delta_\varepsilon^x y, \delta_\varepsilon^x z) \leq \varepsilon d^x(y, z) + \varepsilon \mathcal{O}(\varepsilon) \quad ,$$

$$d(x, \delta_\mu^y z) \leq d(x, y) + d(y, \delta_\mu^y z) \leq d(x, y) + \mu d^y(y, z) + \mu \mathcal{O}(\mu) \quad .$$

These estimates allow us to give the following definition.

**Definition 5.3** *A dilatation structure  $(X, d, \delta)$  is linear if for any  $\varepsilon, \mu \in \Gamma$  such that  $\nu(\varepsilon), \nu(\mu) \in (0, 1]$ , and for any  $x, y, z \in X$  sufficiently closed we have*

$$\delta_\varepsilon^x \delta_\mu^y z = \delta_{\mu}^{\delta_\varepsilon^x y} \delta_\varepsilon^x z \quad .$$

Linear dilatation structures are very particular dilatation structures. The next theorem is theorem 5.7 [4]. It is shown that a linear and strong dilatation structure comes from a normed conical group.

**Theorem 5.4** *Let  $(X, d, \delta)$  be a linear dilatation structure. Then the following two statements are equivalent:*

- (a) *For any  $x \in X$  there is an open neighbourhood  $D \subset X$  of  $x$  such that  $(\overline{D}, d^x, \delta)$  is the same dilatation structure as the dilatation structure of the tangent space of  $(X, d, \delta)$  at  $x$ ;*
- (b) *The dilatation structure is strong (that is satisfies A4).*

## 6 Dilatation structures and inverse semigroups

Here we prove that for dilatation structures linearity is equivalent to a generalization of the statement from corollary 1.3. The result is new for Carnot groups and the proof seems to be new even for vector spaces.

**Definition 6.1** *A dilatation structure  $(X, d, \delta)$  has the Menelaos property if for any two sufficiently closed  $x, y \in X$  and for any  $\varepsilon, \mu \in \Gamma$  with  $\nu(\varepsilon), \nu(\mu) \in (0, 1)$  we have*

$$\delta_\varepsilon^x \delta_\mu^y = \delta_{\varepsilon\mu}^w \quad ,$$

where  $w \in X$  is the fixed point of the contraction  $\delta_\varepsilon^x \delta_\mu^y$  (thus depending on  $x, y$  and  $\varepsilon, \mu$ ).

**Theorem 6.2** *A linear dilatation structure has the Menelaos property.*

**Proof.** Let  $x, y \in X$  be sufficiently closed and  $\varepsilon, \mu \in \Gamma$  with  $\nu(\varepsilon), \nu(\mu) \in (0, 1)$ . We shall define two sequences  $x_n, y_n \in X$ ,  $n \in \mathbb{N}$ .

We begin with  $x_0 = x$ ,  $y_0 = y$ . Let us define by induction

$$x_{n+1} = \delta_\mu^{\delta_\varepsilon^{x_n} y_n} x_n \quad , \quad y_{n+1} = \delta_\varepsilon^{x_n} y_n \quad . \quad (6.0.1)$$

In order to check if the definition is correct we have to prove that for any  $n \in \mathbb{N}$ , if  $x_n, y_n$  are sufficiently closed then  $x_{n+1}, y_{n+1}$  are sufficiently closed too.

Indeed, due to the linearity of the dilatation structure, we can write the first part of (6.0.1) as:

$$x_{n+1} = \delta_\varepsilon^{x_n} \delta_\mu^{y_n} x_n \quad .$$

Then we can estimate the distance between  $x_{n+1}, y_{n+1}$  like this:

$$d(x_{n+1}, y_{n+1}) = d(\delta_\varepsilon^{x_n} \delta_\mu^{y_n} x_n, \delta_\varepsilon^{x_n} y_n) = \nu(\varepsilon) d(\delta_\mu^{y_n} x_n, y_n) = \nu(\varepsilon\mu) d(x_n, y_n) \quad .$$

From  $\nu(\varepsilon\mu) < 1$  it follows that  $d(x_{n+1}, y_{n+1}) < d(x_n, y_n)$ , therefore  $x_{n+1}, y_{n+1}$  are sufficiently closed. We also find out that

$$\lim_{n \rightarrow \infty} d(x_n, y_n) = 0 \quad . \quad (6.0.2)$$

Further we use twice the linearity of the dilatation structure:

$$\delta_\varepsilon^{x_n} \delta_\mu^{y_n} = \delta_\mu^{\delta_\varepsilon^{x_n} y_n} \delta_\varepsilon^{x_n} = \delta_\mu^{\delta_\varepsilon^{\delta_\mu^{x_n} y_n} x_n} \delta_\varepsilon^{\delta_\mu^{x_n} y_n} \quad .$$

By definition (6.0.1) we arrive at the conclusion that for any  $n \in \mathbb{N}$

$$\delta_\varepsilon^{x_n} \delta_\mu^{y_n} = \delta_\varepsilon^x \delta_\mu^y \quad . \quad (6.0.3)$$

From relation (6.0.3) we deduce that the first part of (6.0.1) can be written as:

$$x_{n+1} = \delta_\varepsilon^{x_n} \delta_\mu^{y_n} x_n = \delta_\varepsilon^x \delta_\mu^y x_n \quad .$$

The transformation  $\delta_\varepsilon^x \delta_\mu^y$  is a contraction of coefficient  $\nu(\varepsilon\mu) < 1$ , therefore we easily get:

$$\lim_{n \rightarrow \infty} x_n = w \quad , \quad (6.0.4)$$

where  $w$  is the unique fixed point of the contraction  $\delta_\varepsilon^x \delta_\mu^y$ .

We put together (6.0.2) and (6.0.4) and we get the limit:

$$\lim_{n \rightarrow \infty} y_n = w \quad , \quad (6.0.5)$$

Using relations (6.0.4), (6.0.5), we may pass to the limit with  $n \rightarrow \infty$  in relation (6.0.3):

$$\delta_\varepsilon^x \delta_\mu^y = \lim_{n \rightarrow \infty} \delta_\varepsilon^{x_n} \delta_\mu^{y_n} = \delta_\varepsilon^w \delta_\mu^w = \delta_{\varepsilon\mu}^w \quad .$$

The proof is done.  $\square$

**Corollary 6.3** *Let  $(X, d, \delta)$  be a strong linear dilatation structure, with group  $\Gamma = (0, +\infty)$  and the morphism  $\nu$  equal to identity. Any element of the inverse subsemi-group of  $I(X)$  generated by dilatations is locally a dilatation  $\delta_\varepsilon^x$  or a left translation  $\Sigma^x(y, \cdot)$ .*

**Proof.** Let  $(X, d, \delta)$  be a strong linear dilatation structure. From the linearity and theorem 6.2 we deduce that we have to care only about the results of compositions of two dilatations  $\delta_\varepsilon^x, \delta_\mu^y$ , with  $\varepsilon\mu = 1$ .

The dilatation structure is strong, therefore by theorem 5.4 the dilatation structure is locally coming from a conical group. In a conical group we can make the following computation (here  $\delta_\varepsilon = \delta_\varepsilon^e$  with  $e$  the neutral element of the conical group):

$$\delta_\varepsilon^x \delta_{\varepsilon^{-1}}^y z = x \delta_\varepsilon (x^{-1} y \delta_{\varepsilon^{-1}} (y^{-1} z)) = x \delta_\varepsilon (x^{-1} y) y^{-1} z \quad .$$

Therefore the composition of dilatations  $\delta_\varepsilon^x \delta_\mu^y$ , with  $\varepsilon\mu = 1$ , is a left translation.

Another easy computation shows that composition of left translations with dilatations are dilatations. The proof end by remarking that all the statements are local.  $\square$

## References

- [1] E. Artin, Geometric algebra. Interscience Publishers, Inc., New York-London, (1957), Chapter II, section 2.
- [2] A. Bellaïche, The tangent space in sub-Riemannian geometry, in: Sub-Riemannian Geometry, A. Bellaïche, J.-J. Risler eds., *Progress in Mathematics*, **144**, Birkhäuser, (1996), 4–78
- [3] M. Buliga, Dilatation structures I. Fundamentals, *J. Gen. Lie Theory Appl.*, Vol 1 (2007), No 2, 65-95
- [4] M. Buliga, Contractible groups and linear dilatation structures, (2007), submitted, available as e-print <http://xxx.arxiv.org/abs/0705.1440>
- [5] M. Gromov, Groups of polynomial growth and expanding maps. Inst. Hautes Études Sci. Publ. Math. No. 53, 53–73, (1981)
- [6] J.M. Howie, An introduction to semigroup theory. L.M.S. Monographs, No. 7. Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, (1976), sec. V.1.10
- [7] P. Pansu, Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un, *Annals of Math.*, **129** (1989), 1–60
- [8] E. Siebert, Contractive automorphisms on locally compact groups, *Math. Z.*, 191, 73-90, (1986)